FAKULTET TEHNIČKIH NAUKA
GEODEZIJA I GEOINFORMATIKA

RAČUN IZRAVNANJA
- VEŽBA 6 -

NOVI SAD, 2024.

Izravnanje po metodi uslovnih merenja

- Linearni funkcionalni i stohastički model uslovnog izravnanja

  $\mathbf{A}^T\mathbf{v} + \boldsymbol{\omega} = \mathbf{0}$ – funkcionalni model

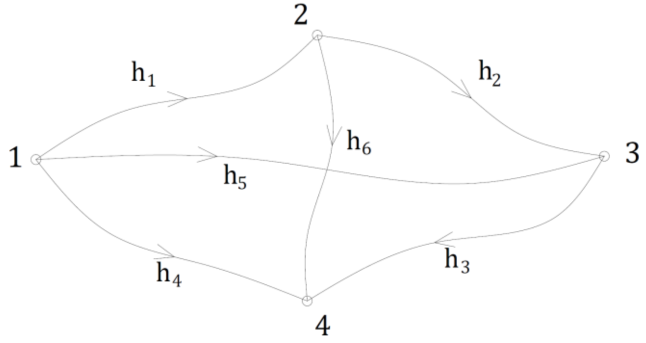
  $\mathbf{P}_l = \mathbf{Q}_l^{-1}, E(\mathbf{v}) = \mathbf{0}$ – stohastički model

- Funkcionalnim modelom definisani su matematički uslovi između merenih veličina u mreži.
- Stohastički model definiše određene pretpostavke u vezi sa merenjima.

Funkcionalni model

- **Primer 1 – jednodimenzionalna geodetska mreža**

Slobodnu nivelmansku mrežu izravnati po metodi uslovnih merenja. *A priori* standard za ocenu nepoznatih parametara iznosi $\sigma_0 = 2$.



Broj nezavisnih matematičkih uslova:

$$r = O + D - 1 = 3 + 1 - 1 = 3$$

O – broj zatvorenih poligona koji su međusobno nezavisni (u sebi sadrže bar jednu visinsku razliku koja nije sadržana u drugim poligonima)
D – broj datih repera

Poligon I: $\hat{h}_1 + \hat{h}_6 - \hat{h}_4 = 0$

Poligon II: $\hat{h}_2 + \hat{h}_3 - \hat{h}_6 = 0$

Poligon III: $\hat{h}_1 + \hat{h}_2 - \hat{h}_5 = 0$

Slobodni članovi:

$$\omega_1 = h_1 + h_6 - h_4$$
$$\omega_2 = h_2 + h_3 - h_6$$
$$\omega_3 = h_1 + h_2 - h_5$$

Jednačine popravaka:

$$v_1 + v_6 - v_4 + \omega_1 = 0$$
$$v_2 + v_3 - v_6 + \omega_2 = 0$$
$$v_1 + v_2 - v_5 + \omega_3 = 0$$

Funkcionalni model

- **Primer 1 – jednodimenzionalna geodetska mreža**

  Formiranje matrice dizajna $\mathbf{A}$ i vektora slobodnih članova $\boldsymbol{\omega}$:

$$\mathbf{A}^T = \begin{array}{cl} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} & \begin{array}{l} \textit{Poligon I} \\ \textit{Poligon II} \\ \textit{Poligon III} \end{array} \end{array} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Napomena:

- Slobodna članove ω_i izraziti u milimetrima.

Stohastički model

- **Primer 1 – jednodimenzionalna geodetska mreža**

Formiranje matrice težina $\mathbf{P}$:

Težine merenja P_{h_i} predstavljaju stepen poverenja u rezultate merenja.

$$P_{h_i} = \frac{1}{S_i},$$

S_i – dužina nivelmanske strane u kilometrima.

$$P_{h_1} = \frac{1}{S_1} \quad P_{h_4} = \frac{1}{S_4}$$

$$P_{h_2} = \frac{1}{S_2} \quad P_{h_5} = \frac{1}{S_5}$$

$$P_{h_3} = \frac{1}{S_3} \quad P_{h_6} = \frac{1}{S_6}$$

$$\mathbf{P} = \begin{array}{c} \begin{bmatrix} P_{h_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{h_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{h_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{h_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{h_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{h_6} \end{bmatrix} \end{array} \begin{array}{c} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{array}$$

Kompletno rešenje zadatka dostupno je u fajlu *Račun izravnanja - Vežba 6.xlsx*.

Primena metoda najmanjih kvadrata (MNK)

- Sistem normalnih jednačina

$$\mathbf{N}\mathbf{k} + \boldsymbol{\omega} = \mathbf{0}$$

$$\mathbf{N} = \mathbf{A}^T\mathbf{P}^{-1}\mathbf{A}$$

- Ocena popravaka merenih veličina

$$\mathbf{k} = -\mathbf{N}^{-1}\boldsymbol{\omega}$$

$$\hat{\mathbf{v}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{k}$$

Kontrola računanja:
$\hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}} \equiv -\mathbf{k}^T\boldsymbol{\omega}$

- Ocena disperzionog koeficijenta

$$m_0^2 = \frac{\hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}}}{r}$$

r – broj nezavisnih matematičkih uslova

Globlani test na grube greške

- Hipoteze

  $H_0 \colon \sigma^2 = \sigma_0^2$ protiv $H_a \colon \sigma^2 \neq \sigma_0^2$,

  pri čemu je $\sigma^2 = M(m_0^2)$, a M operator matematičkog očekivanja.

- Test statistika

  $T = \frac{m_0^2}{\sigma_0^2} \sim F_{1-\alpha,r,\infty}$

  Excel: $F_{1-\alpha,r,\infty}$ → **FINV($\boldsymbol{\alpha}$, $\boldsymbol{r}$,10000)**

  Ukoliko je $T < F_{1-\alpha,r,\infty}$, nulta hipoteza se ne odbacuje, tj. nema grubih grešaka.

  Ukoliko je $T \geq F_{1-\alpha,r,\infty}$, nulta hipoteza se odbacuje, pa konstatujemo da su u merenjima prisutne grube greške.

Izravnate vrednosti merenih veličina

- Izravnate vrednosti visinskih razlika

$$\hat{h}_1 = h_1 + \hat{v}_1$$
$$\hat{h}_2 = h_2 + \hat{v}_2$$
$$\hat{h}_3 = h_3 + \hat{v}_3$$
$$\hat{h}_4 = h_4 + \hat{v}_4$$
$$\hat{h}_5 = h_5 + \hat{v}_5$$
$$\hat{h}_6 = h_6 + \hat{v}_6$$

$$\hat{\mathbf{v}} = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{v}_4 \\ \hat{v}_5 \\ \hat{v}_6 \end{bmatrix}$$

Izravnanje po metodi uslovnih merenja sa nepoznatim parametrima

- Linearni funkcionalni i stohastički model

  $\mathbf{A}^T\mathbf{v} + \mathbf{B}\hat{\mathbf{x}} + \boldsymbol{\omega} = \mathbf{0}$ – funkcionalni model

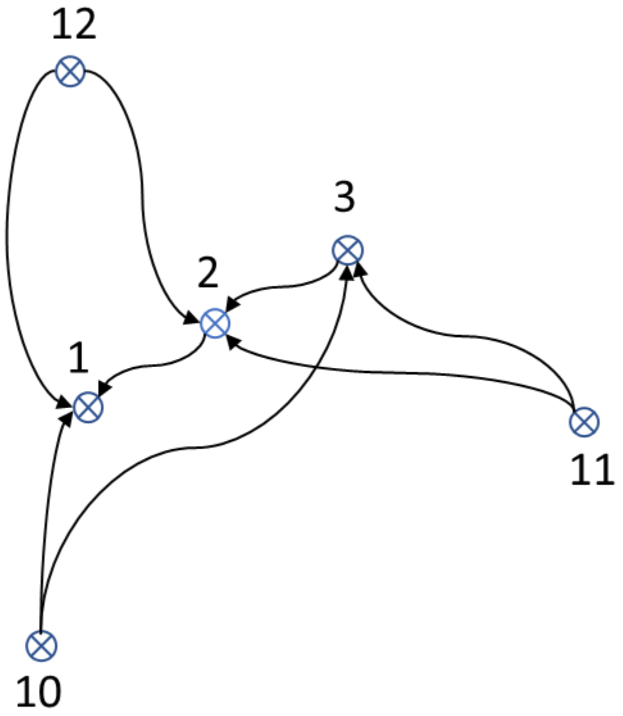
  $\mathbf{P}_l = \mathbf{Q}_l^{-1}, E(\mathbf{v}) = \mathbf{0}$ – stohastički model

- U ovom slučaju merene veličine i nepoznati parametri učestvuju u istim matematičkim uslovima.

Funkcionalni model

- **Primer 2 – jednodimenzionalna geodetska mreža**

U nivelmanskoj mreži koja se sastoji od 6 repera merene su visinske razlike metodom geometrijskog nivelmana. Izravnati rezultate merenja po metodi uslovnih merenja sa nepoznatim parametrima.
Dati reperi: 10, 11 i 12
Nepoznati reperi: 1, 2 i 3



Približne visine repera:

$$H_{1,1}^0 = H_{10} + \Delta h_{10-1}, H_{1,2}^0 = H_{12} + \Delta h_{12-1}, H_1^0 = \frac{H_{1,1}^0 + H_{1,2}^0}{2}$$

$$H_{2,1}^0 = H_{11} + \Delta h_{11-2}, H_{2,2}^0 = H_{12} + \Delta h_{12-2}, H_2^0 = \frac{H_{2,1}^0 + H_{2,2}^0}{2}$$

$$H_{3,1}^0 = H_{10} + \Delta h_{10-3}, H_{3,2}^0 = H_{11} + \Delta h_{11-3}, H_3^0 = \frac{H_{3,1}^0 + H_{3,2}^0}{2}$$

Funkcionalni model

- **Primer 2 – jednodimenzionalna geodetska mreža**

  Matematički uslovi:

$$H_{10} + \Delta h_{10-1} + v_{\Delta h_{10-1}} = H_1^0 + dH_1$$
$$H_{10} + \Delta h_{10-3} + v_{\Delta h_{10-3}} = H_3^0 + dH_3$$
$$H_{11} + \Delta h_{11-2} + v_{\Delta h_{11-2}} = H_2^0 + dH_2$$
$$H_{11} + \Delta h_{11-3} + v_{\Delta h_{11-3}} = H_3^0 + dH_3$$
$$H_{12} + \Delta h_{12-1} + v_{\Delta h_{12-1}} = H_1^0 + dH_1$$
$$H_{12} + \Delta h_{12-2} + v_{\Delta h_{12-2}} = H_2^0 + dH_2$$
$$H_2^0 + dH_2 + \Delta h_{2-1} + v_{\Delta h_{2-1}} = H_1^0 + dH_1$$
$$H_3^0 + dH_3 + \Delta h_{3-2} + v_{\Delta h_{3-2}} = H_2^0 + dH_2$$

$$v_{\Delta h_{10-1}} - dH_1 + \omega_{\Delta h_{10-1}} = 0$$
$$v_{\Delta h_{10-3}} - dH_3 + \omega_{\Delta h_{10-3}} = 0$$
$$v_{\Delta h_{11-2}} - dH_2 + \omega_{\Delta h_{11-2}} = 0$$
$$v_{\Delta h_{11-3}} - dH_3 + \omega_{\Delta h_{11-3}} = 0$$
$$v_{\Delta h_{12-1}} - dH_1 + \omega_{\Delta h_{12-1}} = 0$$
$$v_{\Delta h_{12-2}} - dH_2 + \omega_{\Delta h_{12-2}} = 0$$
$$v_{\Delta h_{2-1}} + dH_2 - dH_1 + \omega_{\Delta h_{2-1}} = 0$$
$$v_{\Delta h_{3-2}} + dH_3 - dH_2 + \omega_{\Delta h_{3-2}} = 0$$

$$\omega_{\Delta h_{10-1}} = H_{10} + \Delta h_{10-1} - H_1^0$$
$$\omega_{\Delta h_{10-3}} = H_{10} + \Delta h_{10-3} - H_3^0$$
$$\omega_{\Delta h_{11-2}} = H_{11} + \Delta h_{11-2} - H_2^0$$
$$\omega_{\Delta h_{11-3}} = H_{11} + \Delta h_{11-3} - H_3^0$$
$$\omega_{\Delta h_{12-1}} = H_{12} + \Delta h_{12-1} - H_1^0$$
$$\omega_{\Delta h_{12-2}} = H_{12} + \Delta h_{12-2} - H_2^0$$
$$\omega_{\Delta h_{2-1}} = H_2^0 + \Delta h_{2-1} - H_1^0$$
$$\omega_{\Delta h_{3-2}} = H_3^0 + \Delta h_{3-2} - H_2^0$$

Funkcionalni model

- **Primer 2 – jednodimenzionalna geodetska mreža**

Formiranje matrice dizajna $\mathbf{A}$, matrice $\mathbf{B}$ i vektora slobodnih članova $\boldsymbol{\omega}$:

$$\mathbf{A}^T = \begin{array}{c} \begin{matrix} v_{\Delta h_{10-1}} & & & \cdots & & & & v_{\Delta h_{3-2}} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{B} = \begin{array}{c} \begin{matrix} dH_1 & dH_2 & dH_3 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_{\Delta h_{10-1}} \\ \omega_{\Delta h_{10-3}} \\ \omega_{\Delta h_{11-2}} \\ \omega_{\Delta h_{11-3}} \\ \omega_{\Delta h_{12-1}} \\ \omega_{\Delta h_{12-2}} \\ \omega_{\Delta h_{2-1}} \\ \omega_{\Delta h_{3-2}} \end{bmatrix}$$

Napomena:
Slobodna članove ω_i izraziti u milimetrima.

Stohastički model

- **Primer 2 – jednodimenzionalna geodetska mreža**

Formiranje težina merenja:

$$P_{\Delta h_{i-j}} = \frac{1}{S_{i-j}},$$

S_{i-j} – dužina nivelmanske strane u kilometrima.

$$P_{\Delta h_{10-1}} = \frac{1}{S_{10-1}}, P_{\Delta h_{12-1}} = \frac{1}{S_{12-1}}$$

$$P_{\Delta h_{10-3}} = \frac{1}{S_{10-3}}, P_{\Delta h_{12-2}} = \frac{1}{S_{12-2}}$$

$$P_{\Delta h_{11-2}} = \frac{1}{S_{11-2}}, P_{\Delta h_{2-1}} = \frac{1}{S_{2-1}}$$

$$P_{\Delta h_{11-3}} = \frac{1}{S_{11-3}}, P_{\Delta h_{3-2}} = \frac{1}{S_{3-2}}$$

$$\mathbf{P} = \begin{array}{cc} \begin{bmatrix} P_{\Delta h_{10-1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{\Delta h_{10-3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{\Delta h_{11-2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{\Delta h_{11-3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{\Delta h_{12-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{12-2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{2-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{3-2}} \end{bmatrix} & \begin{array}{c} \Delta h_{10-1} \\ \Delta h_{10-3} \\ \Delta h_{11-2} \\ \Delta h_{11-3} \\ \Delta h_{12-1} \\ \Delta h_{12-2} \\ \Delta h_{2-1} \\ \Delta h_{3-2} \end{array} \end{array}$$

Kompletno rešenje primera 2 dostupno je u fajlu *Račun izravnanja - Vežba 6.xlsx*.

Primena metoda najmanjih kvadrata (MNK)

- Sistem normalnih jednačina

$$\mathbf{N}\mathbf{k} + \mathbf{B}\hat{\mathbf{x}} + \boldsymbol{\omega} = \mathbf{0}$$
$$\mathbf{B}^T\mathbf{k} = \mathbf{0}$$
$$\mathbf{N} = \mathbf{A}^T\mathbf{P}^{-1}\mathbf{A}$$

- Ocena nepoznatih parametara i popravaka merenih veličina

$$\mathbf{k} = [\mathbf{N}^{-1}\mathbf{B}(\mathbf{B}^T\mathbf{N}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{N}^{-1} - \mathbf{N}^{-1}]\boldsymbol{\omega}$$
$$\hat{\mathbf{x}} = -(\mathbf{B}^T\mathbf{N}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{N}^{-1}\boldsymbol{\omega}$$
$$\hat{\mathbf{v}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{k}$$

Kontrola računanja:
$$\hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}} \equiv -\mathbf{k}^T\boldsymbol{\omega}$$

- Ocena disperzionog koeficijenta

$$m_0^2 = \frac{\hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}}}{r - u}$$

r – broj matematičkih uslova
u – broj nepoznatih parametara

Izravnate vrednosti merenja i nepoznatih parametara

- Izravnate vrednosti nepoznatih parametara

$$\hat{H}_1 = H_1^0 + \widehat{dH}_1$$
$$\hat{H}_2 = H_2^0 + \widehat{dH}_2$$
$$\hat{H}_3 = H_3^0 + \widehat{dH}_3$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \widehat{dH}_1 \\ \widehat{dH}_2 \\ \widehat{dH}_3 \end{bmatrix}$$

- Izravnate vrednosti merenih veličina

$$\widehat{\Delta h}_{10-1} = \Delta h_{10-1} + \hat{v}_{\Delta h_{10-1}}$$
$$\widehat{\Delta h}_{10-3} = \Delta h_{10-3} + \hat{v}_{\Delta h_{10-3}}$$
$$\widehat{\Delta h}_{11-2} = \Delta h_{11-2} + \hat{v}_{\Delta h_{11-2}}$$
$$\widehat{\Delta h}_{11-3} = \Delta h_{11-3} + \hat{v}_{\Delta h_{11-3}}$$
$$\widehat{\Delta h}_{12-1} = \Delta h_{12-1} + \hat{v}_{\Delta h_{12-1}}$$
$$\widehat{\Delta h}_{12-2} = \Delta h_{12-2} + \hat{v}_{\Delta h_{12-2}}$$
$$\widehat{\Delta h}_{2-1} = \Delta h_{2-1} + \hat{v}_{\Delta h_{2-1}}$$
$$\widehat{\Delta h}_{3-2} = \Delta h_{3-2} + \hat{v}_{\Delta h_{3-2}}$$

$$\hat{\mathbf{v}} = \begin{bmatrix} \hat{v}_{\Delta h_{10-1}} \\ \hat{v}_{\Delta h_{10-3}} \\ \hat{v}_{\Delta h_{11-2}} \\ \hat{v}_{\Delta h_{11-3}} \\ \hat{v}_{\Delta h_{12-1}} \\ \hat{v}_{\Delta h_{12-2}} \\ \hat{v}_{\Delta h_{2-1}} \\ \hat{v}_{\Delta h_{3-2}} \end{bmatrix}$$