

FAKULTET TEHNIČKIH NAUKA  
GEODEZIJA I GEOINFORMATIKA

RAČUN IZRAVNANJA  
- VEŽBA 6 -

NOVI SAD, 2025.

# Izravnanje po metodi uslovnih merenja

- Linearni funkcionalni i stohastički model uslovnog izravnanja

$$\mathbf{A}^T \mathbf{v} + \boldsymbol{\omega} = \mathbf{0} \quad \text{– funkcionalni model}$$

$$\mathbf{P}_l = \mathbf{Q}_l^{-1}, E(\mathbf{v}) = \mathbf{0} \quad \text{– stohastički model}$$

- Funkcionalnim modelom definisani su matematički uslovi između merenih veličina u mreži.
- Stohastički model definiše određene pretpostavke u vezi sa merenjima.

# Funkcionalni model

## • Primer 1 – jednodimenzionalna geodetska mreža

Slobodnu nivelmansku mrežu izravnati po metodi uslovnih merenja. *A priori* standard za ocenu nepoznatih parametara iznosi  $\sigma_0 = 2$ .

Broj nezavisnih matematičkih uslova:

$$r = O + D - 1 = 3 + 1 - 1 = 3$$

$O$  – broj zatvorenih poligona koji su međusobno nezavisni (u sebi sadrže bar jednu visinsku razliku koja nije sadržana u drugim poligonima)

$D$  – broj datih repera

Poligon I:  $\hat{h}_1 + \hat{h}_6 - \hat{h}_4 = 0$

Poligon II:  $\hat{h}_2 + \hat{h}_3 - \hat{h}_6 = 0$

Poligon III:  $\hat{h}_1 + \hat{h}_2 - \hat{h}_5 = 0$

Slobodni članovi:

$$\omega_1 = h_1 + h_6 - h_4$$

$$\omega_2 = h_2 + h_3 - h_6$$

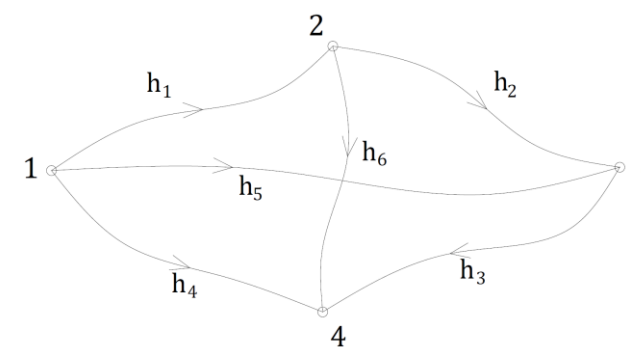
$$\omega_3 = h_1 + h_2 - h_5$$

Jednačine popravaka:

$$v_1 + v_6 - v_4 + \omega_1 = 0$$

$$v_2 + v_3 - v_6 + \omega_2 = 0$$

$$v_1 + v_2 - v_5 + \omega_3 = 0$$



# Funkcionalni model

- **Primer 1 – jednodimenzionalna geodetska mreža**

Formiranje matrice dizajna  $\mathbf{A}$  i vektora slobodnih članova  $\boldsymbol{\omega}$ :

$$\mathbf{A}^T = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} & \begin{matrix} \textit{Poligon I} \\ \textit{Poligon II} \\ \textit{Poligon III} \end{matrix} \end{matrix} & \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Napomena:

- Slobodna članove  $\omega_i$  izraziti u milimetrima.

# Stohastički model

- **Primer 1 – jednodimenzionalna geodetska mreža**

Formiranje matrice težina  $\mathbf{P}$ :

Težine merenja  $P_{h_i}$  predstavljaju stepen poverenja u rezultate merenja.

$$P_{h_i} = \frac{1}{S_i},$$

$S_i$  – dužina nivelmanske strane u kilometrima.

$$P_{h_1} = \frac{1}{S_1} \quad P_{h_4} = \frac{1}{S_4}$$

$$P_{h_2} = \frac{1}{S_2} \quad P_{h_5} = \frac{1}{S_5}$$

$$P_{h_3} = \frac{1}{S_3} \quad P_{h_6} = \frac{1}{S_6}$$

$$\mathbf{P} = \begin{bmatrix} P_{h_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{h_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{h_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{h_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{h_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{h_6} \end{bmatrix} \begin{matrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{matrix}$$

**Kompletno rešenje zadatka dostupno je u fajlu *Račun izravnjanja - Vežba 6.xlsx*.**

# Primena metoda najmanjih kvadrata (MNK)

- Sistem normalnih jednačina

$$\mathbf{Nk} + \boldsymbol{\omega} = \mathbf{0}$$

$$\mathbf{N} = \mathbf{A}^T \mathbf{P}^{-1} \mathbf{A}$$

- Ocena popravaka merenih veličina

$$\mathbf{k} = -\mathbf{N}^{-1} \boldsymbol{\omega}$$

$$\hat{\mathbf{v}} = \mathbf{P}^{-1} \mathbf{A} \mathbf{k}$$

Kontrola računanja:

$$\hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}} \equiv -\mathbf{k}^T \boldsymbol{\omega}$$

- Ocena disperzionog koeficijenta

$$m_0^2 = \frac{\hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}}}{r} \quad r - \text{broj nezavisnih matematičkih uslova}$$

# Globlani test na grube greške

- Hipoteze

$$H_0: \sigma^2 = \sigma_0^2 \text{ protiv } H_a: \sigma^2 \neq \sigma_0^2,$$

pri čemu je  $\sigma^2 = M(m_0^2)$ , a  $M$  operator matematičkog očekivanja.

- Test statistika

$$T = \frac{m_0^2}{\sigma_0^2} \sim F_{1-\alpha, r, \infty}$$

**Excel:**  $F_{1-\alpha, r, \infty} \rightarrow \text{FINV}(\alpha, r, 10000)$

Ukoliko je  $T < F_{1-\alpha, r, \infty}$ , nulta hipoteza se ne odbacuje, tj. nema grubih grešaka.

Ukoliko je  $T \geq F_{1-\alpha, r, \infty}$ , nulta hipoteza se odbacuje, pa konstatujemo da su u merenjima prisutne grube greške.

# Izravnate vrednosti merenih veličina

- Izravnate vrednosti visinskih razlika

$$\hat{h}_1 = h_1 + \hat{v}_1$$

$$\hat{h}_2 = h_2 + \hat{v}_2$$

$$\hat{h}_3 = h_3 + \hat{v}_3$$

$$\hat{h}_4 = h_4 + \hat{v}_4$$

$$\hat{h}_5 = h_5 + \hat{v}_5$$

$$\hat{h}_6 = h_6 + \hat{v}_6$$

$$\hat{\mathbf{v}} = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{v}_4 \\ \hat{v}_5 \\ \hat{v}_6 \end{bmatrix}$$

# Izravnanje po metodi uslovnih merenja sa nepoznatim parametrima

- Linearni funkcionalni i stohastički model

$$\mathbf{A}^T \mathbf{v} + \mathbf{B}\hat{\mathbf{x}} + \boldsymbol{\omega} = \mathbf{0} \quad - \text{funkcionalni model}$$

$$\mathbf{P}_l = \mathbf{Q}_l^{-1}, E(\mathbf{v}) = \mathbf{0} \quad - \text{stohastički model}$$

- U ovom slučaju merene veličine i nepoznati parametri učestvuju u istim matematičkim uslovima.

# Funkcionalni model

## • Primer 2 – jednodimenzionalna geodetska mreža

U nivelmanskoj mreži koja se sastoji od 6 repera merene su visinske razlike metodom geometrijskog nivelmana. Izravnati rezultate merenja po metodi uslovnih merenja sa nepoznatim parametrima.

Dati reperi: 10, 11 i 12

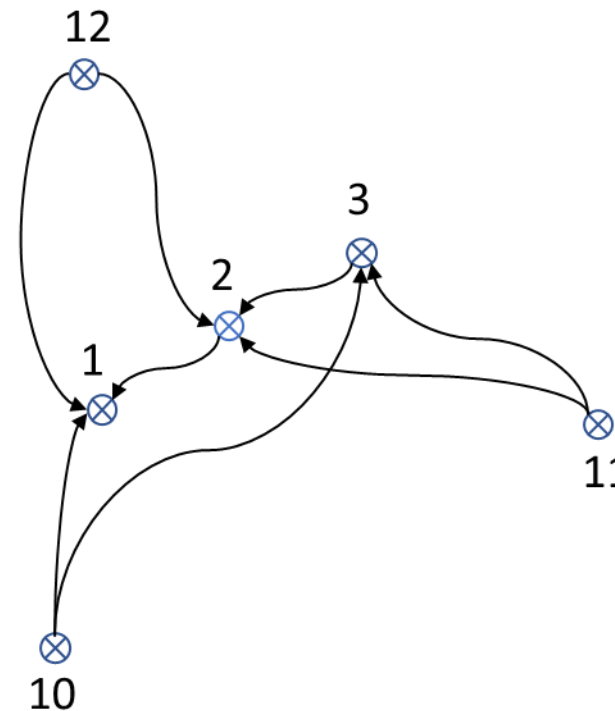
Nepoznati reperi: 1, 2 i 3

Približne visine repera:

$$H_{1,1}^0 = H_{10} + \Delta h_{10-1}, H_{1,2}^0 = H_{12} + \Delta h_{12-1}, H_1^0 = \frac{H_{1,1}^0 + H_{1,2}^0}{2}$$

$$H_{2,1}^0 = H_{11} + \Delta h_{11-2}, H_{2,2}^0 = H_{12} + \Delta h_{12-2}, H_2^0 = \frac{H_{2,1}^0 + H_{2,2}^0}{2}$$

$$H_{3,1}^0 = H_{10} + \Delta h_{10-3}, H_{3,2}^0 = H_{11} + \Delta h_{11-3}, H_3^0 = \frac{H_{3,1}^0 + H_{3,2}^0}{2}$$



# Funkcionalni model

- **Primer 2 – jednodimenzionalna geodetska mreža**

Matematički uslovi:

$H_{10} + \Delta h_{10-1} + v_{\Delta h_{10-1}} = H_1^0 + dH_1$	$v_{\Delta h_{10-1}} - dH_1 + \omega_{\Delta h_{10-1}} = 0$	$\omega_{\Delta h_{10-1}} = H_{10} + \Delta h_{10-1} - H_1^0$
$H_{10} + \Delta h_{10-3} + v_{\Delta h_{10-3}} = H_3^0 + dH_3$	$v_{\Delta h_{10-3}} - dH_3 + \omega_{\Delta h_{10-3}} = 0$	$\omega_{\Delta h_{10-3}} = H_{10} + \Delta h_{10-3} - H_3^0$
$H_{11} + \Delta h_{11-2} + v_{\Delta h_{11-2}} = H_2^0 + dH_2$	$v_{\Delta h_{11-2}} - dH_2 + \omega_{\Delta h_{11-2}} = 0$	$\omega_{\Delta h_{11-2}} = H_{11} + \Delta h_{11-2} - H_2^0$
$H_{11} + \Delta h_{11-3} + v_{\Delta h_{11-3}} = H_3^0 + dH_3$	$v_{\Delta h_{11-3}} - dH_3 + \omega_{\Delta h_{11-3}} = 0$	$\omega_{\Delta h_{11-3}} = H_{11} + \Delta h_{11-3} - H_3^0$
$H_{12} + \Delta h_{12-1} + v_{\Delta h_{12-1}} = H_1^0 + dH_1$	$v_{\Delta h_{12-1}} - dH_1 + \omega_{\Delta h_{12-1}} = 0$	$\omega_{\Delta h_{12-1}} = H_{12} + \Delta h_{12-1} - H_1^0$
$H_{12} + \Delta h_{12-2} + v_{\Delta h_{12-2}} = H_2^0 + dH_2$	$v_{\Delta h_{12-2}} - dH_2 + \omega_{\Delta h_{12-2}} = 0$	$\omega_{\Delta h_{12-2}} = H_{12} + \Delta h_{12-2} - H_2^0$
$H_2^0 + dH_2 + \Delta h_{2-1} + v_{\Delta h_{2-1}} = H_1^0 + dH_1$	$v_{\Delta h_{2-1}} + dH_2 - dH_1 + \omega_{\Delta h_{2-1}} = 0$	$\omega_{\Delta h_{2-1}} = H_2^0 + \Delta h_{2-1} - H_1^0$
$H_3^0 + dH_3 + \Delta h_{3-2} + v_{\Delta h_{3-2}} = H_2^0 + dH_2$	$v_{\Delta h_{3-2}} + dH_3 - dH_2 + \omega_{\Delta h_{3-2}} = 0$	$\omega_{\Delta h_{3-2}} = H_3^0 + \Delta h_{3-2} - H_2^0$

# Funkcionalni model

- **Primer 2 – jednodimenzionalna geodetska mreža**

Formiranje matrice dizajna **A**, matrice **B** i vektora slobodnih članova  $\omega$ :

$$\mathbf{A}^T = \begin{matrix} & v_{\Delta h_{10-1}} & & \dots & & v_{\Delta h_{3-2}} & & \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} & \mathbf{B} = \begin{matrix} & dH_1 & dH_2 & dH_3 \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{matrix} & \boldsymbol{\omega} = \begin{matrix} \begin{bmatrix} \omega_{\Delta h_{10-1}} \\ \omega_{\Delta h_{10-3}} \\ \omega_{\Delta h_{11-2}} \\ \omega_{\Delta h_{11-3}} \\ \omega_{\Delta h_{12-1}} \\ \omega_{\Delta h_{12-2}} \\ \omega_{\Delta h_{2-1}} \\ \omega_{\Delta h_{3-2}} \end{bmatrix} \end{matrix}$$

Napomena:

Slobodna članove  $\omega_i$  izraziti u milimetrima.

# Stohastički model

- **Primer 2 – jednodimenzionalna geodetska mreža**

Formiranje težina merenja:

$$P_{\Delta h_{i-j}} = \frac{1}{S_{i-j}},$$

$S_{i-j}$  – dužina nivelmanske strane u kilometrima.

$$P_{\Delta h_{10-1}} = \frac{1}{S_{10-1}}, P_{\Delta h_{12-1}} = \frac{1}{S_{12-1}}$$

$$P_{\Delta h_{10-3}} = \frac{1}{S_{10-3}}, P_{\Delta h_{12-2}} = \frac{1}{S_{12-2}}$$

$$P_{\Delta h_{11-2}} = \frac{1}{S_{11-2}}, P_{\Delta h_{2-1}} = \frac{1}{S_{2-1}}$$

$$P_{\Delta h_{11-3}} = \frac{1}{S_{11-3}}, P_{\Delta h_{3-2}} = \frac{1}{S_{3-2}}$$

$$\mathbf{P} = \begin{bmatrix} P_{\Delta h_{10-1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{\Delta h_{10-3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{\Delta h_{11-2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{\Delta h_{11-3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{\Delta h_{12-1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{12-2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{2-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{\Delta h_{3-2}} & 0 \end{bmatrix} \begin{bmatrix} \Delta h_{10-1} \\ \Delta h_{10-3} \\ \Delta h_{11-2} \\ \Delta h_{11-3} \\ \Delta h_{12-1} \\ \Delta h_{12-2} \\ \Delta h_{2-1} \\ \Delta h_{3-2} \end{bmatrix}$$

Kompletno rešenje primera 2 dostupno je u fajlu *Račun izravnjanja - Vežba 6.xlsx*.

# Primena metoda najmanjih kvadrata (MNK)

- Sistem normalnih jednačina

$$\mathbf{N}\mathbf{k} + \mathbf{B}\hat{\mathbf{x}} + \boldsymbol{\omega} = \mathbf{0}$$

$$\mathbf{B}^T \mathbf{k} = \mathbf{0}$$

$$\mathbf{N} = \mathbf{A}^T \mathbf{P}^{-1} \mathbf{A}$$

- Ocena nepoznatih parametara i popravaka merenih veličina

$$\mathbf{k} = [\mathbf{N}^{-1} \mathbf{B} (\mathbf{B}^T \mathbf{N}^{-1} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{N}^{-1} - \mathbf{N}^{-1}] \boldsymbol{\omega}$$

$$\hat{\mathbf{x}} = -(\mathbf{B}^T \mathbf{N}^{-1} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{N}^{-1} \boldsymbol{\omega}$$

$$\hat{\mathbf{v}} = \mathbf{P}^{-1} \mathbf{A} \mathbf{k}$$

Kontrola računanja:

$$\hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}} \equiv -\mathbf{k}^T \boldsymbol{\omega}$$

- Ocena disperzionog koeficijenta

$$m_0^2 = \frac{\hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}}}{r - u}$$

$r$  – broj matematičkih uslova

$u$  – broj nepoznatih parametara

# Izravnote vrednosti merjenja i nepoznatih parametara

- Izravnote vrednosti nepoznatih parametara

$$\begin{aligned}\widehat{H}_1 &= H_1^0 + d\widehat{H}_1 \\ \widehat{H}_2 &= H_2^0 + d\widehat{H}_2 \\ \widehat{H}_3 &= H_3^0 + d\widehat{H}_3\end{aligned}\quad \widehat{\mathbf{x}} = \begin{bmatrix} d\widehat{H}_1 \\ d\widehat{H}_2 \\ d\widehat{H}_3 \end{bmatrix}$$

- Izravnote vrednosti merenih veličina

$$\begin{aligned}\widehat{\Delta h}_{10-1} &= \Delta h_{10-1} + \widehat{v}_{\Delta h_{10-1}} \\ \widehat{\Delta h}_{10-3} &= \Delta h_{10-3} + \widehat{v}_{\Delta h_{10-3}} \\ \widehat{\Delta h}_{11-2} &= \Delta h_{11-2} + \widehat{v}_{\Delta h_{11-2}} \\ \widehat{\Delta h}_{11-3} &= \Delta h_{11-3} + \widehat{v}_{\Delta h_{11-3}} \\ \widehat{\Delta h}_{12-1} &= \Delta h_{12-1} + \widehat{v}_{\Delta h_{12-1}} \\ \widehat{\Delta h}_{12-2} &= \Delta h_{12-2} + \widehat{v}_{\Delta h_{12-2}} \\ \widehat{\Delta h}_{2-1} &= \Delta h_{2-1} + \widehat{v}_{\Delta h_{2-1}} \\ \widehat{\Delta h}_{3-2} &= \Delta h_{3-2} + \widehat{v}_{\Delta h_{3-2}}\end{aligned}\quad \widehat{\mathbf{v}} = \begin{bmatrix} \widehat{v}_{\Delta h_{10-1}} \\ \widehat{v}_{\Delta h_{10-3}} \\ \widehat{v}_{\Delta h_{11-2}} \\ \widehat{v}_{\Delta h_{11-3}} \\ \widehat{v}_{\Delta h_{12-1}} \\ \widehat{v}_{\Delta h_{12-2}} \\ \widehat{v}_{\Delta h_{2-1}} \\ \widehat{v}_{\Delta h_{3-2}} \end{bmatrix}$$